

Separation of Coupled Systems of Schrodinger Equations by Darboux transformations

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Abstract

Darboux transformations in one independent variable have found numerous applications in various field of mathematics and physics. In this paper we show that the extension of these transformations to two dimensions can be used to decouple systems of Schrodinger equations and provide explicit representation for three classes of such systems. We show also that there is an elegant relationship between these transformations and analytic complex matrix functions.

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1 Introduction

For many years Darboux transformations in one independent variable have found numerous applications in various field of mathematics and physics [1-17]. (Refs. [1,2] contain an extensive list of references). In particular the factorization method and its generalizations [13-14] which have been instrumental in many physical applications (including SUSY QM [15]) has its roots based on these transformations. Recently however these transformations were generalized and applied to systems of nonlinear equations such as the KdV hierarchy and soliton solutions [2,5,7]. In addition various applications of this method in geometry were worked out and form an important ongoing research area [2]. Extensions of the method to multidimensional oriented Riemann manifolds [8], time dependent potentials [9,16] and shape invariant potentials [17] have appeared in the literature.

It is surprising that in spite of this extensive research effort the theory and applications of these transformations in two variables and its elegant relationship to complex analytic function theory has not been recognized widely in the literature.[4]

Here is a short overview of Darboux transformations for Schrodinger equation in one variable. (In the following we assume that all functions under consideration are smooth).

We say that the solutions of two Schrodinger equations with different potentials $u(x), v(x)$ i.e.

$$\phi'' = (u(x) + \lambda)\phi \quad (1.1)$$

$$\psi'' = (v(x) + \lambda)\psi. \quad (1.2)$$

are related by a Darboux transformation if there exist $A(x), B(x)$ so that

$$\psi = \left[A(x) + B(x) \frac{\partial}{\partial x} \right] \phi(x). \quad (1.3)$$

Letting $B(x) = 1$ one can easily show that in order for eqs (1.1), (1.2) to be related by the transformation (1.3) $A(x), u(x), v(x)$ must satisfy;

$$A'' + u' + A(u - v) = 0 \quad (1.4)$$

$$2A' + u - v = 0 \quad (1.5)$$

Eliminating $(u - v)$ between these equations and integration yields

$$A' - A^2 + u = -\nu \quad (1.6)$$

where ν is an integration constant. Eq. (1.6) is a Riccati equation which can be linearized by the transformation $A = -\zeta'/\zeta$ which leads to

$$\zeta'' = (u(x) + \nu)\zeta. \quad (1.7)$$

Thus ζ is an eigenfunction of the original eq. (1.2) with $\lambda = \nu$. From (1.5) we now infer that

$$v = u - 2(\ln \zeta)'' \quad (1.8)$$

i.e. a Darboux transformation changes the potential function $u(x)$ by $\Delta u = -2(\ln \zeta)''$ where ζ is an arbitrary eigenfunction of (1.1).

From a more general point of view we note that Darboux transformation for equations in real variables form a special case of the Laplace-Darboux transformations for the complex variable linear equation [2]

$$\Psi_{z\bar{z}} + a\Psi_z + b\Psi_{\bar{z}} + \mathbf{V}\Psi = \mathbf{0} \quad (1.9)$$

(where $z = x + iy$).

Our objective in this paper is to introduce a novel application of these transformations to decouple systems of partial differential equations in two dimensions of the form

$$\nabla^2 \Psi + \mathbf{V}(x, y)\Psi = \mathbf{0} \quad (1.10)$$

where $\Psi^T = (\psi_1(x, y), \psi_2(x, y))$ and $\mathbf{V}(x, y)$ is a 2×2 matrix. Such systems appear frequently in many applications of classical and modern physics [4,6].

We note that the more general form of this equation

$$\mathbf{A}(x, y)\nabla^2 \Psi + \mathbf{V}(x, y)\Psi = \mathbf{0} \quad (1.11)$$

can be transformed into (1.10) if $\mathbf{A}(x, y)$ is (locally) invertible.

The plan of the paper is as follows: In Sec 2 we derive the basic equations that constrain Darboux transformations in two dimensions. In Secs. 3, 4, 5 we solve these equations explicitly for three different classes of the potential matrix \mathbf{V} and end up in Sec. 6 with summary and conclusions.

2 Darboux transformations in two Dimensions.

We shall say that two systems of partial differential equations (PDEs) in two independent variables

$$\nabla^2 \Phi + \mathbf{U}(x, y)\Phi = \mathbf{0} \quad (2.1)$$

$$\nabla^2 \Psi + \mathbf{V}(x, y)\Psi = \mathbf{0} \quad (2.2)$$

are related by a Darboux transformation if there exist 2×2 nonsingular matrices with smooth function entries $\mathbf{C}_1(x, y)$, $\mathbf{C}_2(x, y)$, $\mathbf{C}_3(x, y)$ so that their solutions satisfy

$$\Psi = \left[\mathbf{C}_1(x, y) + \mathbf{C}_2(x, y)\frac{\partial}{\partial x} + \mathbf{C}_3(x, y)\frac{\partial}{\partial y} \right] \Phi. \quad (2.3)$$

For brevity we drop in the following the dependence of the various functions on the independent variables.

Using eq. (2.3) to substitute for Ψ in eq. (2.2) and eliminating the higher order derivatives of Φ , $\nabla^2 \Phi$ and $\frac{\partial^2 \Phi}{\partial y^2}$ using eq. (2.1) we obtain

$$\begin{aligned}
& 2 \left[\frac{\partial \mathbf{C}_2}{\partial y} - \frac{\partial \mathbf{C}_3}{\partial x} \right] \frac{\partial^2 \Phi}{\partial x^2} + 2 \left[\frac{\partial \mathbf{C}_2}{\partial y} + \frac{\partial \mathbf{C}_3}{\partial x} \right] \frac{\partial^2 \Phi}{\partial x \partial y} + \\
& \left\{ \nabla^2 \mathbf{C}_3 + 2 \frac{\partial \mathbf{C}_1}{\partial y} - \mathbf{C}_3 \mathbf{U} + \mathbf{V} \mathbf{C}_3 \right\} \frac{\partial \Phi}{\partial y} + \left\{ \nabla^2 \mathbf{C}_2 + 2 \frac{\partial \mathbf{C}_1}{\partial x} - \mathbf{C}_2 \mathbf{U} + \mathbf{V} \mathbf{C}_2 \right\} \frac{\partial \Phi}{\partial x} + \\
& \left\{ \nabla^2 \mathbf{C}_1 - 2 \frac{\partial \mathbf{C}_3}{\partial y} \mathbf{U} - \mathbf{C}_2 \frac{\partial \mathbf{U}}{\partial x} - \mathbf{C}_3 \frac{\partial \mathbf{U}}{\partial y} - \mathbf{C}_1 \mathbf{U} + \mathbf{V} \mathbf{C}_1 \right\} \Phi = 0
\end{aligned} \tag{2.4}$$

To satisfy this equation we treat Φ and its derivatives as independent variables and let their coefficients be zero. This leads then to the following system of equations.

$$\frac{\partial \mathbf{C}_2}{\partial x} - \frac{\partial \mathbf{C}_3}{\partial y} = 0 \tag{2.5}$$

$$\frac{\partial \mathbf{C}_2}{\partial y} + \frac{\partial \mathbf{C}_3}{\partial x} = 0 \tag{2.6}$$

$$\nabla^2 \mathbf{C}_2 + 2 \frac{\partial \mathbf{C}_1}{\partial x} - \mathbf{C}_2 \mathbf{U} + \mathbf{V} \mathbf{C}_2 = 0 \tag{2.7}$$

$$\nabla^2 \mathbf{C}_3 + 2 \frac{\partial \mathbf{C}_1}{\partial y} - \mathbf{C}_3 \mathbf{U} + \mathbf{V} \mathbf{C}_3 = 0 \tag{2.8}$$

$$\nabla^2 \mathbf{C}_1 - 2 \frac{\partial \mathbf{C}_3}{\partial y} \mathbf{U} - \mathbf{C}_2 \frac{\partial \mathbf{U}}{\partial x} - \mathbf{C}_3 \frac{\partial \mathbf{U}}{\partial y} - \mathbf{C}_1 \mathbf{U} + \mathbf{V} \mathbf{C}_1 = 0 \tag{2.9}$$

We observe that eq. (2.9) can be rewritten in a symmetric form in $\mathbf{C}_2, \mathbf{C}_3$ in view of eq.(2.5).

Eqs.(2.5),(2.6) are Cauchy-Riemann equations for $\mathbf{C}_2, \mathbf{C}_3$. Hence the corresponding entries in this matrices must be harmonic conjugates and the entries of

$$\mathbf{D} = \mathbf{C}_2 + i \mathbf{C}_3 \tag{2.10}$$

are analytic. In view of this fact $\nabla^2 \mathbf{C}_2 = \nabla^2 \mathbf{C}_3 = 0$ and eqs. (2.7),(2.8) simplify to

$$2 \frac{\partial \mathbf{C}_1}{\partial x} - \mathbf{C}_2 \mathbf{U} + \mathbf{V} \mathbf{C}_2 = 0, \quad 2 \frac{\partial \mathbf{C}_1}{\partial y} - \mathbf{C}_3 \mathbf{U} + \mathbf{V} \mathbf{C}_3 = 0. \tag{2.11}$$

From (2.11) we have

$$\mathbf{V} = [\mathbf{C}_2 \mathbf{U} - 2 \frac{\partial \mathbf{C}_1}{\partial x}] \mathbf{C}_2^{-1}, \quad \mathbf{V} = [\mathbf{C}_3 \mathbf{U} - 2 \frac{\partial \mathbf{C}_1}{\partial y}] \mathbf{C}_3^{-1}. \tag{2.12}$$

Hence

$$2 \left[\frac{\partial \mathbf{C}_1}{\partial x} \mathbf{C}_2^{-1} - \frac{\partial \mathbf{C}_1}{\partial y} \mathbf{C}_3^{-1} \right] = \mathbf{C}_2 \mathbf{U} \mathbf{C}_2^{-1} - \mathbf{C}_3 \mathbf{U} \mathbf{C}_3^{-1} \quad (2.13)$$

Using (2.12) to eliminate \mathbf{V} from (2.9) leads to

$$\nabla^2 \mathbf{C}_1 - 2 \frac{\partial \mathbf{C}_3}{\partial y} \mathbf{U} - \mathbf{C}_2 \frac{\partial \mathbf{U}}{\partial \mathbf{x}} - \mathbf{C}_3 \frac{\partial \mathbf{U}}{\partial \mathbf{y}} - \mathbf{C}_1 \mathbf{U} + [\mathbf{C}_3 \mathbf{U} - 2 \frac{\partial \mathbf{C}_1}{\partial \mathbf{y}}] \mathbf{C}_3^{-1} \mathbf{C}_1 = \mathbf{0} \quad (2.14)$$

Assuming that $\mathbf{C}_2, \mathbf{C}_3$ were chosen already to satisfy (2.5)- (2.6) the system (2.13)-(2.14) consists of eight coupled nonlinear equations for the entries of \mathbf{C}_1 and \mathbf{U} . However in order to decouple the system (2.2) by the transformation (2.3) the resulting \mathbf{U} must be diagonal or upper (lower) triangular. In the following we assume that the desired form of the matrix \mathbf{U} is diagonal viz.

$$\mathbf{U} = \begin{pmatrix} u_{11}(x, y) & 0 \\ 0 & u_{22}(x, y) \end{pmatrix}. \quad (2.15)$$

Under this constraint the system (2.13)-(2.14) is an over determined system of eight coupled nonlinear equations in six (unknown) functions.

To derive solutions for these equations our strategy will be as follows: First we choose a proper form for the entries of the matrices \mathbf{C}_2 and \mathbf{C}_3 . Then we solve (2.13)-(2.14) (under some restrictions) for \mathbf{C}_1 and \mathbf{U} . Finally we obtain \mathbf{V} from (2.12).

In the next three sections we provide solutions and explicit examples for this decoupling procedure for different choices of the matrices \mathbf{C}_2 and \mathbf{C}_3

3 \mathbf{C}_2 and \mathbf{C}_3 are Matrices with Constant Entries

When \mathbf{C}_2 and \mathbf{C}_3 are matrices with real constants entries viz.

$$\mathbf{C}_2 = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \quad \mathbf{C}_3 = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}. \quad (3.1)$$

equations (2.5), (2.6) are satisfied by default. We also assume that \mathbf{C}_1 is of the form

$$\mathbf{C}_1 = \begin{pmatrix} c_{11}(x, y) & c_{12}(x, y) \\ c_{21}(x, y) & c_{22}(x, y) \end{pmatrix}. \quad (3.2)$$

In this setting eq. (2.13) leads to a coupled system of equations which can be simplified if we assume that

$$e_{21} = \frac{d_{21}e_{11}}{d_{11}}, \quad e_{12} = \frac{d_{12}e_{22}}{d_{22}} \quad (3.3)$$

The resulting equations for the entries of the matrix \mathbf{C}_1 can be solved (after some algebra) and we find that the general solution for these entries is

$$c_{11} = F_{11}(w_1), \quad c_{12} = F_{12}(w_2), \quad c_{21} = F_{21}(w_1), \quad c_{22} = F_{22}(w_2), \quad (3.4)$$

where

$$w_1 = d_{11}x + e_{11}y, \quad w_2 = e_{22}y + d_{22}x,$$

and F_{ij} are smooth functions of the indicated variable. To proceed we assume a special form of the functions F_{ij} , $i, j = 1, 2$

$$\begin{aligned} F_{11} &= h_{11}(d_{11}x + e_{11}y), \quad F_{12} = \frac{h_{22}d_{12}(d_{22}x + e_{22}y)}{d_{22}}, \\ F_{21} &= \frac{h_{11}d_{21}(d_{11}x + e_{11}y)}{d_{11}}, \quad F_{22} = h_{22}(d_{22}x + e_{22}y) \end{aligned} \quad (3.5)$$

where h_{ij} are constants. Using these expressions to evaluate the right hand side of

$$\mathbf{E} = \nabla^2 \mathbf{C}_1 - 2 \frac{\partial \mathbf{C}_1}{\partial y} \mathbf{C}_3^{-1} \mathbf{C}_1. \quad (3.6)$$

We find:

$$\mathbf{E} = \begin{pmatrix} -2h_{11}^2(d_{11}x + e_{11}y) & \frac{-2h_{22}^2d_{12}(d_{22}x + e_{22}y)}{d_{22}} \\ \frac{-2h_{11}^2d_{21}(d_{11}x + e_{11}y)}{d_{11}} & -2h_{22}^2(d_{22}x + e_{22}y) \end{pmatrix}. \quad (3.7)$$

Substituting this result in (2.14) and using (3.3) we obtain only two independent equations for u_{11} and u_{22}

$$d_{11} \frac{\partial u_{11}(x, y)}{\partial x} + e_{11} \frac{\partial u_{11}(x, y)}{\partial y} + 2h_{11}^2(d_{11}x + e_{11}y) = 0, \quad (3.8)$$

$$d_{22} \frac{\partial u_{22}(x, y)}{\partial x} + e_{22} \frac{\partial u_{22}(x, y)}{\partial y} + 2h_{22}^2(d_{22}x + e_{22}y) = 0. \quad (3.9)$$

The general solution to these equations is

$$u_{11} = \frac{h_{11}^2}{2} \frac{e_{11}^2(e_{11}^2 - d_{11}^2)x^2 + d_{11}^2(d_{11}^2 - e_{11}^2)y^2 - 2d_{11}e_{11}(d_{11}^2 + e_{11}^2)xy}{d_{11}^2e_{11}^2} + H_1\left(\frac{d_{11}y - e_{11}x}{d_{11}}\right) \quad (3.10)$$

$$u_{22} = \frac{h_{22}^2}{2} \frac{e_{22}^2(e_{22}^2 - d_{22}^2)x^2 + d_{22}^2(d_{22}^2 - e_{22}^2)y^2 - 2d_{22}e_{22}(d_{22}^2 + e_{22}^2)xy}{d_{22}^2e_{22}^2} + H_2\left(\frac{d_{22}y - e_{22}x}{d_{22}}\right) \quad (3.11)$$

where H_1 and H_2 are arbitrary (smooth) functions of the indicated variables.

The general form of the matrix \mathbf{V} (from (2.12)) is

$$\mathbf{V} = \frac{1}{\det(\mathbf{C}_2)} \begin{pmatrix} d_{11}d_{22}(u_{11} - 2h_{11}) - d_{12}d_{21}(u_{22} - 2h_{22}) & d_{11}d_{12}(u_{22} - u_{11} + 2(h_{11} - h_{22})) \\ d_{22}d_{21}(u_{11} - u_{22} + 2(h_{22} - h_{11})) & d_{21}d_{12}(u_{11} - 2h_{11}) - d_{11}d_{22}(u_{22} - 2h_{22}) \end{pmatrix} \quad (3.12)$$

where $\det(\mathbf{C}_2)$ is the determinant of \mathbf{C}_2 .

Example: Let the matrices C_2 and C_3 be chosen as

$$\mathbf{C}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{C}_3 = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}. \quad (3.13)$$

Furthermore let $h_{11} = h_{22} = 1$ and (for simplicity) $H_1 = H_2 = 0$. The potential matrix V for the coupled system becomes

$$\mathbf{V} = -\frac{1}{4} \begin{pmatrix} -3x^2 + \frac{3}{4}y^2 + 9xy + 8 & -3x^2 + \frac{3}{4}y^2 + xy \\ 3x^2 - \frac{3}{4}y^2 - xy & 3x^2 - \frac{3}{4}y^2 - 9xy - 8 \end{pmatrix}. \quad (3.14)$$

Thus the coupled system represents two coupled two dimensional oscillators. For the resulting uncoupled system we have

$$u_{11} = \frac{1}{2} \left(3x^2 - \frac{3}{4}y^2 - 5xy \right), \quad u_{22} = -2xy. \quad (3.15)$$

4 The Matrix $D = z^n C$, where C has Constant Entries

In this case we assume that the matrices \mathbf{C}_2 and \mathbf{C}_3 are of the form

$$\mathbf{C}_2 = \text{Re}(z^n) \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \quad \mathbf{C}_3 = \text{Im}(z^n) \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}. \quad (4.1)$$

where e_{ij}, d_{ij} are real and $\text{Re}(z^n), \text{Im}(z^n)$ are the real and imaginary parts of z^n , $0 < n$. (Similar treatment can be made if we replace z^n by iz^n). In view of (2.5)-(2.6) we must have also $e_{ij} = d_{ij}$.

Since in polar coordinates $\text{Re}(z^n) = r^n \cos(n\theta)$ and $\text{Im}(z^n) = r^n \sin(n\theta)$ it is expedient to work in this coordinate system.

It is easy to see that in this setting the right hand side of (2.13) is zero and we obtain the following equation for the the elements c_{ij} of \mathbf{C}_1

$$\sin((n-1)\theta) \frac{\partial c_{ij}}{\partial r} - \frac{\cos((n-1)\theta)}{r} \frac{\partial c_{ij}}{\partial \theta} = 0, \quad i, j = 1, 2. \quad (4.2)$$

The general solution of this equation is $c_{ij} = F_{ij}(w)$ where

$$w(r, t) = \frac{\cos((n-1)\theta)}{r^{n-1}} \quad (4.3)$$

and F_{ij} are smooth functions of w . However in order to obtain a consistent and decoupled system of equations from (2.14) we shall assume the following special form of \mathbf{C}_1

$$\mathbf{C}_1 = \begin{pmatrix} F_{11} & \frac{d_{12}F_{22}}{d_{22}} \\ \frac{d_{21}F_{11}}{d_{11}} & F_{22} \end{pmatrix}. \quad (4.4)$$

With these choices for the elements of \mathbf{C}_1 the matrix \mathbf{E} has only two independent entries E_{11} and E_{22} . (To derive this result we used (4.3)).

$$\mathbf{E}_{ii} = \frac{1}{r^{2n}} [F'_{ii} + \frac{1}{d_{ii}} F_{ii}^2]' = \frac{h_{ii}}{r^{2n}} \quad (4.5)$$

where primes denote differentiation with respect to w . The matrix \mathbf{E} takes the following form:

$$\mathbf{E} = \frac{1}{r^{2n}} \begin{pmatrix} h_{11} & \frac{d_{12}h_{22}}{d_{22}} \\ \frac{d_{21}h_{11}}{d_{11}} & h_{22} \end{pmatrix}. \quad (4.6)$$

where $h_{ii} = h_{ii}(w)$. Substituting these results in (2.14) and using (4.3) we find that this system also have only two independent equations

$$r^n \cos((n-1)\theta) \frac{\partial u_{ii}}{\partial r} + r^{n-1} \sin((n-1)\theta) \frac{\partial u_{ii}}{\partial \theta} + 2nr^{n-1} \cos((n-1)\theta) u_{ii} = \frac{h_{ii}}{d_{ii}r^{2n}}. \quad (4.7)$$

If we choose F_{ii} so that $h_{ii} = 0$ then the general solution of (4.7) is

$$u_{ii} = \frac{1}{r^{2n}} H_{ii} \left(\frac{\sin((n-1)\theta)}{r^{n-1}} \right) \quad (4.8)$$

where H_{ii} are smooth functions of the indicated variable. The general form of the matrix \mathbf{V} in this case is

$$V_{11} = \frac{1}{\det(D)} \left[d_{11}d_{22}u_{11} - d_{12}d_{21}u_{22} + 4r^{-2n} \left(d_{22}F'_{11} - \frac{d_{21}d_{12}}{d_{22}} F'_{22} \right) \right] \quad (4.9)$$

$$V_{12} = \frac{1}{\det(D)} \left[d_{11}d_{12}u_{11} - d_{11}d_{12}u_{22} + 4r^{-2n} \left(d_{12}F'_{11} - \frac{d_{11}d_{12}}{d_{22}} F'_{22} \right) \right] \quad (4.10)$$

$$V_{21} = \frac{1}{\det(D)} \left[d_{21}d_{22}u_{11} - d_{21}d_{22}u_{22} + 4r^{-2n} \left(\frac{d_{21}d_{22}}{d_{11}} F'_{11} - d_{21}F'_{22} \right) \right] \quad (4.11)$$

$$V_{22} = \frac{1}{\det(D)} \left[d_{21}d_{12}u_{11} - d_{11}d_{22}u_{22} + 4r^{-2n} \left(\frac{d_{21}d_{12}F'_{11}}{d_{11}} - d_{11}F'_{22} \right) \right] \quad (4.12)$$

where $\det(D) = d_{11}d_{22} - d_{12}d_{21}$.

We present two examples.

Example: Let $n = 1$ and

$$\mathbf{C} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (4.13)$$

Furthermore to satisfy (4.5) we let $F_{11} = F_{22} = \frac{1}{r}$. For these choices we find that

$$u_{11} = \frac{A}{r^2}, \quad u_{22} = \frac{B}{r^2} \quad (4.14)$$

where A, B are constants. The corresponding matrix potential for the coupled system is

$$\mathbf{V} = \begin{pmatrix} \frac{A+B}{2r^2} - \frac{4}{r^6} & \frac{A-B}{2r^2} \\ -\frac{A-B}{2r^2} & -\frac{A+B}{2r^2} + \frac{4}{r^6} \end{pmatrix}. \quad (4.15)$$

We observe that in this case the potentials are radial (no dependence on θ).

Example: Let $n = 2$ and choose C as in the previous example. To satisfy (4.5) with $h_{ii} = 0$ we choose $F_{11} = F_{22} = \frac{r}{r+\cos\theta}$ and let

$$H_{11} = \frac{A \sin \theta}{r}, \quad H_{22} = \frac{B \sin \theta}{r}. \quad (4.16)$$

With these choices we have

$$u_{11} = \frac{A \sin \theta}{r^5}, \quad u_{22} = \frac{B \sin \theta}{r^5}. \quad (4.17)$$

The corresponding matrix potential for the coupled system is

$$\mathbf{V} = \begin{pmatrix} \frac{(A+B) \sin \theta}{2r^5} - \frac{4}{r^2(r+\cos \theta)^2} & \frac{(A-B) \sin \theta}{2r^5} \\ -\frac{(A-B) \sin \theta}{2r^5} & -\frac{(A+B) \sin \theta}{2r^5} + \frac{4}{r^2(r+\cos \theta)^2} \end{pmatrix}. \quad (4.18)$$

5 \mathbf{C}_2 and \mathbf{C}_3 have Orthogonal Columns

In this case we assume that \mathbf{C}_2 and \mathbf{C}_3 have the following structure

$$\mathbf{C}_2 = \begin{pmatrix} f_{11}(x, y) & f_{22}(x, y) \\ -f_{11}(x, y) & f_{22}(x, y) \end{pmatrix}, \quad \mathbf{C}_3 = \begin{pmatrix} g_{11}(x, y) & g_{22}(x, y) \\ -g_{11}(x, y) & g_{22}(x, y) \end{pmatrix}. \quad (5.1)$$

where f_{ii} and g_{ii} are complex conjugate. We assume also that the desired form of \mathbf{U} is given by (2.15). Using (3.2) to define \mathbf{C}_1 , equation (2.13) yields after some algebra

$$f_{11}(x, y) \frac{\partial c_{k1}}{\partial y} - g_{11}(x, y) \frac{\partial c_{k1}}{\partial x} = 0, \quad k = 1, 2, \quad (5.2)$$

$$f_{22}(x, y) \frac{\partial c_{k2}}{\partial y} - g_{22}(x, y) \frac{\partial c_{k2}}{\partial x} = 0, \quad k = 1, 2. \quad (5.3)$$

This leads us to consider the following equations

$$f_{11}(x, y) dx + g_{11}(x, y) dy = 0, \quad f_{22}(x, y) dx + g_{22}(x, y) dy = 0. \quad (5.4)$$

Although these equations are not exact an integrating factor for these equations is given by

$$\frac{1}{f_{11}^2(x, y) + g_{11}^2(x, y)}$$

and

$$\frac{1}{f_{22}^2(x, y) + g_{22}^2(x, y)}$$

respectively. (This fact follows from Cauchy-Riemann equations for \mathbf{C}_2 and \mathbf{C}_3). The general solution of these equations can be expressed therefore by the standard formulas

$$w_i(x, y) = \int_{x_0}^x \frac{f_{ii}(x, y)}{f_{ii}^2(x, y) + g_{ii}^2(x, y)} dx + \int_{y_0}^y \frac{g_{ii}(x_0, y)}{f_{ii}(x_0, y)^2 + g_{ii}(x_0, y)^2} dy, \quad i = 1, 2. \quad (5.5)$$

It follows then that the general solution for the entries of the matrix \mathbf{C}_1 is

$$c_{11} = F_{11}(w_1), \quad c_{12} = F_{12}(w_2), \quad c_{21} = F_{21}(w_1), \quad c_{22} = F_{22}(w_2). \quad (5.6)$$

However in this case we shall let $F_{12}(w_2) = F_{22}(w_2)$ and $F_{21}(w_1) = -F_{11}(w_1)$ i.e the matrix \mathbf{C}_1 is of the form

$$\mathbf{C}_1 = \begin{pmatrix} F_{11}(w_1) & F_{22}(w_2) \\ -F_{11}(w_1) & F_{22}(w_2) \end{pmatrix}. \quad (5.7)$$

The resulting matrix \mathbf{E} is the form

$$\mathbf{E} = \begin{pmatrix} h_{11}(x, y) & h_{22}(x, y) \\ -h_{11}(x, y) & h_{22}(x, y) \end{pmatrix}. \quad (5.8)$$

where h_{ii} are defined by

$$h_{11} = \frac{F_{11}(w_1)'' - 2F_{11}(w_1)F_{11}(w_1)'}{(f_{11}^2 + g_{11}^2)}, \quad (5.9)$$

$$h_{22} = \frac{F_{22}(w_2)'' - 2F_{22}(w_2)F_{22}(w_2)'}{(f_{22}^2 + g_{22}^2)}. \quad (5.10)$$

Substituting these results in (2.14) we obtain (only) two independent equations for u_{ii} .

$$g_{ii} \frac{\partial u_{ii}}{\partial y} + f_{ii} \frac{\partial u_{ii}}{\partial x} + 2 \frac{\partial g_{ii}}{\partial y} u_{ii} = h_{ii}, \quad i = 1, 2. \quad (5.11)$$

Using (2.12) we find that the elements of the matrix \mathbf{V} are

$$\mathbf{V}_{11} = \mathbf{V}_{22} = \frac{1}{2}(u_{11} + u_{22}) - \frac{F'_{11}}{(f_{11}^2 + g_{11}^2)} - \frac{F'_{22}}{(f_{22}^2 + g_{22}^2)}, \quad (5.12)$$

$$\mathbf{V}_{12} = \mathbf{V}_{21} = \frac{1}{2}(u_{22} - u_{11}) + \frac{F'_{11}}{(f_{11}^2 + g_{11}^2)} - \frac{F'_{22}}{(f_{22}^2 + g_{22}^2)}. \quad (5.13)$$

We now consider the particular case where

$$\mathbf{C}_2 = \begin{pmatrix} a \operatorname{Re}(z^n) & b \operatorname{Re}(z^m) \\ -a \operatorname{Re}(z^n) & b \operatorname{Re}(z^m) \end{pmatrix}, \quad \mathbf{C}_3 = \begin{pmatrix} a \operatorname{Im}(z^n) & b \operatorname{Im}(z^m) \\ -a \operatorname{Im}(z^n) & b \operatorname{Im}(z^m) \end{pmatrix}. \quad (5.14)$$

where a, b are real and $n, m > 0$, $m \neq n$. (When $n = m$ this case reduces to the situation discussed in the previous section).

As in the previous section it is expedient to work this example using polar coordinates. Eqs (5.2) reduces to (4.2) and (5.3) reduces to the same equation with n being replaced by m . The general form of the matrix C_1 is given by (5.7) where

$$w_1 = \frac{\cos((n-1)\theta)}{r^{n-1}}, \quad w_2 = \frac{\cos((m-1)\theta)}{r^{m-1}}. \quad (5.15)$$

Setting $h_{11} = h_{22} = 0$ we obtain from (5.9),(5.10) that

$$F_{11} = \frac{\tan(\frac{w_1+c_2}{c_1})}{c_1}, \quad \text{or } F_{11} = \text{constant}, \quad (5.16)$$

$$F_{22} = \frac{\tan(\frac{w_2+c_4}{c_3})}{c_3}, \quad \text{or } F_{22} = \text{constant}, \quad (5.17)$$

where c_i , $i = 1 \dots 4$ are constants. Eqs (5.11) for $i = 1, 2$ then reduce to (4.7) (with n being replaced by m for $i = 2$) and therefore (using (4.8))

$$u_{11} = \frac{1}{r^{2n}} H_{11} \left(\frac{\sin((n-1)\theta)}{r^{n-1}} \right), \quad (5.18)$$

$$u_{22} = \frac{1}{r^{2m}} H_{22} \left(\frac{\sin((m-1)\theta)}{r^{m-1}} \right), \quad (5.19)$$

where H_{11} , H_{22} are smooth functions of the indicated variables. The explicit general form of the matrix potential V for the coupled system in this case is given by (5.12)-(5.13).

If we let $n = 2$, $m = 1$, $F_{ii} = \text{constant}$, $i = 1, 2$ and choose

$$H_{11} = \frac{A \sin \theta}{r}, \quad H_{22} = B \quad (5.20)$$

where A, B are constants then the matrix potential V is given by

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} \frac{A \sin \theta}{r^5} + \frac{B}{r^2} & \frac{B}{r^2} - \frac{A \sin \theta}{r^5} \\ \frac{B}{r^2} - \frac{A \sin \theta}{r^5} & \frac{A \sin \theta}{r^5} + \frac{B}{r^2} \end{pmatrix}. \quad (5.21)$$

6 Summary and Conclusions.

In this paper we showed that Darboux transformations in two dimensions can be applied to decouple systems of PDES. We demonstrated also the close affinity of these transformations to complex analytic functions. Although we were unable to solve (2.13)-(2.14) in general we were able to provide explicit solutions for three classes of equations. Even if the decoupled system is not integrable analytically its numerical solution is less demanding computationally. Furthermore the decoupled system might provide insights about the solutions of the original system which are obvious directly.

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